Not-for-Publication Appendix to Intellectual Property Protection and Innovation: An Inverted-U Relationship

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This document will provide a formal and detailed proofs for Theorem 1 and Proposition 1 in Furukawa (2010).[1]

1 Proof for Theorem 1

Define $\hat{\phi} \equiv (b^{-1}\alpha^{\frac{1+\alpha-\alpha\psi}{(1-\alpha)(1-\psi)}}(1-\alpha)-\rho)^{-1}$. The theorem is:

Theorem 1 *If and only if $\phi > \hat{\phi} > 0$, the BGP is unique, saddle-path stable, and characterized by the condition for market equilibrium stationarity,*

$$H^* = \varpi_1 (\phi g^* + 1)/\phi + \varpi_1 \rho,$$

and the condition for human capital accumulation stationarity,

$$H^* = \varpi_2 \left((1-\alpha^{\frac{1}{1-\alpha}})n^* + \alpha^{\frac{1}{1-\alpha}} \right)^{\frac{\psi}{1-\psi}}$$

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[1]Furukawa, Yuichi, 2010. "Intellectual Property Protection and Innovation: An Inverted-U Relationship," *Economics Letters*.

plus one side condition $n^* = \frac{1}{\phi g^*+1}$.

Proof.

The equilibrium conditions making up the model are the following equations:

$$Y_t = (S_t L)^{1-\alpha} \int_0^{N_t} x_t(j)^\alpha, \;\; \alpha \in (0,1); \tag{1}$$

$$V_t = \int_t^\infty e^{-(R_\tau - R_t)-t/\phi} \pi_t dt; \tag{2}$$

$$\dot{N}_t^C = \phi^{-1}(N_t - N_t^C); \tag{3}$$

$$H_t = S_t L = \hat{\theta} \int_{-\infty}^t e^{-\theta(t-\tau)} \left(\frac{X_\tau}{N_\tau}\right)^\psi d\tau; \tag{4}$$

$$Y_t = C_t + X_t + b\dot{N}_t; \tag{5}$$

$$\frac{\dot{C}_t}{C_t} = r_t - \rho. \tag{6}$$

The profit maximization condition for the final good firms is given by:

$$p_t(j) = \alpha (S_t L)^{1-\alpha} x_t(j)^{\alpha-1}. \tag{7}$$

Since producing a unit of intermediate goods requires a unit of capital goods, intermediate goods are priced as follows:

$$p_t(j) = \begin{cases} 1, & j \in [0, N_t^C] \\ \frac{1}{\alpha} & j \in [N_t^C, N_t] \end{cases}, \tag{8}$$

noting that the price elasticity is equal to $\frac{1}{1-\alpha}$ by (7). By (7) and (8), the amounts of intermediate goods are given by

$$x_t(j) = \begin{cases} \alpha^{\frac{1}{1-\alpha}} S_t L \equiv x_t^C, & j \in [0, N_t^C] \\ \alpha^{\frac{2}{1-\alpha}} S_t L \equiv x_t^M & j \in [N_t^C, N_t] \end{cases}, \tag{9}$$

and the profit of a monopolistic (i.e., yet-to-be-imitated) intermediate good producer is given by

$$\pi_t(j) = \frac{1-\alpha}{\alpha} x_t^M = \alpha^{\frac{1+\alpha}{1-\alpha}} (1-\alpha) S_t L \equiv \pi_t. \tag{10}$$

Aggregate variables, Y_t and $X_t = \int_0^{N_t} x_t(j)dj$, can be easily caliculated from (1) and (9):

$$\begin{aligned} Y_t &= (S_t L)^{1-\alpha} \int_0^{N_t} x_t(j)^\alpha & (11) \\ &= \alpha^{\frac{\alpha}{1-\alpha}} S_t L \left((1 - \alpha^{\frac{\alpha}{1-\alpha}}) N_t^C + \alpha^{\frac{\alpha}{1-\alpha}} N_t \right); \end{aligned}$$

$$\begin{aligned} X_t &= \int_0^{N_t} x_t(j)dj & (12) \\ &= \alpha^{\frac{1}{1-\alpha}} S_t L \left((1 - \alpha^{\frac{1}{1-\alpha}}) N_t^C + \alpha^{\frac{1}{1-\alpha}} N_t \right). \end{aligned}$$

The market clearing condition, (5), can be rewritten as

$$\frac{\dot{N}_t}{N_t} = \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha) b^{-1} S_t L \left((1 - \alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha)) \frac{N_t^C}{N_t} + \alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha) \right) - \frac{C_t}{bN_t}. \tag{13}$$

By differentiating (2) with respect to t, I can have

$$r_t V_t = \pi_t + \dot{V}_t - \phi^{-1} V_t. \tag{14}$$

Noting that the cost of making an innovation is b is equal to the value, V_t, in equilibrium, this Bellman equation can derive:

$$r_t = \alpha^{\frac{1+\alpha}{1-\alpha}} (1-\alpha) b^{-1} S_t L - \phi^{-1}, \tag{15}$$

noting (10). From (6) and (15), and noting $H_t = S_t L$, the following growth rates are obtained:

$$\frac{\dot{C}_t}{C_t} = \alpha^{\frac{1+\alpha}{1-\alpha}} (1-\alpha) b^{-1} H_t - \phi^{-1} - \rho, \tag{16}$$

$$\frac{\dot{N}_t}{N_t} = \alpha^{\frac{\alpha}{1-\alpha}} (1-\alpha) b^{-1} H_t \left((1 - \alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha)) \frac{N_t^C}{N_t} + \alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha) \right) - \frac{C_t}{bN_t}. \tag{17}$$

$$\frac{\dot{N}_t^C}{N_t^C} = \phi^{-1} \left(\frac{N_t}{N_t^C} - 1 \right), \tag{18}$$

$$\frac{\dot{H}_t}{H_t} = \hat{\theta} \alpha^{\frac{\psi}{1-\alpha}} \left((1 - \alpha^{\frac{1}{1-\alpha}}) \frac{N_t^C}{N_t} + \alpha^{\frac{1}{1-\alpha}} \right)^{\psi} H_t^{\psi-1} - \theta.[2] \tag{19}$$

[2]This can be derived by differentiating () with t, noting (X).

Define $c_t \equiv \frac{C_t}{bN_t}$, $n_t \equiv \frac{N_t^C}{N_t}$, and $h_t \equiv \frac{S_t L}{b}$. Then, equilibrium dynamics can be characterized by the following 3×3 system.

$$\frac{\dot{c}_t}{c_t} = \alpha^{\frac{1+\alpha}{1-\alpha}}(1-\alpha)h_t - \phi^{-1} - \rho - g_t, \tag{20}$$

$$\frac{\dot{n}_t}{n_t} = \phi^{-1}\left(n_t^{-1} - 1\right) - g_t, \tag{21}$$

$$\frac{\dot{h}_t}{h_t} = \hat{\theta}\alpha^{\frac{\psi}{1-\alpha}}\left((1-\alpha^{\frac{1}{1-\alpha}})n_t + \alpha^{\frac{1}{1-\alpha}}\right)^{\psi} b^{\psi-1}h_t^{\psi-1} - \theta, \tag{22}$$

where

$$g_t \equiv \frac{\dot{N}_t}{N_t} = \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)h_t\left((1-\alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha))n_t + \alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha)\right) - c_t. \tag{23}$$

In steady state, $\dot{c}_t = \dot{n}_t = \dot{h}_t = 0$. Denote by c^*, n^*, and h^* the steady-state values of c_t, n_t, and h_t. Noting $\frac{\dot{N}_t}{N_t} = \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)h^*\left((1-\alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha))n^* + \alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha)\right) - c^*$ along the BGP, it is easy to derive the following condition from $\dot{c}_t = 0$:

$$H^* = \frac{\varpi_1(\phi g^* + 1)}{\phi} + \varpi_1\rho, \tag{24}$$

where $g^* = \frac{\dot{N}^*}{N^*} = \frac{\dot{C}^*}{C^*}$ is the rate of innovation, $H^* = bh^*$ is the stock of human capital in steady state equilibrium, and $\varpi_1 \equiv b/\alpha^{\frac{1+\alpha}{1-\alpha}}(1-\alpha)$. *Equation (24) represents the market equilibrium stationarity, i.e.,* $\frac{\dot{N}}{N} = \frac{\dot{C}}{C}$. There are positive relationships of g^* to H^* and ϕ, which are explained by curve ι in Figure 1 of Furukawa (2010).

From $\dot{h}_t = 0$ (with $\dot{n}_t = 0$),

$$H^* = \varpi_2\left((1-\alpha^{\frac{1}{1-\alpha}})n^* + \alpha^{\frac{1}{1-\alpha}}\right)^{\frac{\psi}{1-\psi}} \text{ with } n^* = \frac{1}{\phi g^* + 1} \tag{25}$$

in which $\varpi_2 \equiv \left(\frac{\hat{\theta}\alpha^{\frac{\psi}{1-\alpha}}}{\theta}\right)^{\frac{1}{1-\psi}}$. *Equation (25) represents the human capital accumulation staionarity, i.e.,* $\frac{\dot{H}}{H}$. There is a positive relationship between the fraction of competitive sectors, n^*, and H^*, shown as curve η in Figure 1 of Furukawa (2010). The intuition is explained as follows, which corresponds to Remark 1 of Furukawa (2010). *The accumulation of human capital stock, $\dot{H}_t$, is governed by the average (per variety) use of intermediate goods for final good firms,*

$\frac{X_t}{N_t} = \frac{1}{N_t}\int_0^{N_t} x_t(j)dj$. *The average use of intermediate goods,* $\frac{X_t}{N_t}$*, is an increasing function of the fraction of competitive sectors,* n^**, because competitive sectors supply more intermediate goods than monopolistic sectors,* $x^C > x^M$ *(due to the usual monopolistic pricing). Consequently, there is a positive relation between* n^* *and* H^**.*

Noting $n^* = \frac{1}{\phi g^*+1}$ (from $\dot{n}_t = 0$), the fraction competitive sectors, n^*, decreases with a higher rate of innovation, i.e., a higher g^*, (since innovations introduce more monopolistic firms into the economy) and n^* also decreases with the strength of intellectual property rights (IPR) protection, i.e., a larger ϕ (since stronger IPR protection decreases imitation activities, which play a role to convert monopolistic sectors to competitive ones in the model). Equation (25) can be rewritten as

$$H^* = \varpi_2 \left(\frac{\alpha^{\frac{1}{1-\alpha}}\phi g^* + 1}{\phi g^* + 1} \right)^{\frac{\psi}{1-\psi}}. \tag{26}$$

See Figure 1, in which (24) and (26) are described as upward sloping and downward sloping curves I and L, respectively. The following fact follows.

Claim 1 *The balanced growth path is uniquely determined by a system of equations, (24) and (26), corresponding to the intersection* E *of curves* I *and* L *in Figure 1, if and only if*

$$b^{-1}\alpha^{\frac{1+\alpha-\alpha\psi}{(1-\alpha)(1-\psi)}}(1-\alpha)\left(\frac{\hat{\theta}}{\theta}\right)^{\frac{1}{1-\psi}} > \rho$$

and

$$\phi > \left(b^{-1}\alpha^{\frac{1+\alpha-\alpha\psi}{(1-\alpha)(1-\psi)}}(1-\alpha)\left(\frac{\hat{\theta}}{\theta}\right)^{\frac{1}{1-\psi}} - \rho \right)^{-1}. \tag{27}$$

[3]

Next, the stability of the system will be discussed. It is easy to show that the log-linearized system of original system (20)–(22) has the following coefficient matrix:

$$B = \begin{pmatrix} c^* & -\zeta_1 & \zeta_2 - \zeta_3 \\ c^* & -\zeta_4 - \zeta_1 & -\zeta_3 \\ 0 & \zeta_5 & -\zeta_6 \end{pmatrix},$$

[3] This can be easily derived from $\varpi_1(\phi^{-1} + \rho) < \varpi_2$.

where

$$
\begin{aligned}
\xi_1 &\equiv \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)(1-\alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha))h^*n^* > 0,\\
\zeta_2 &\equiv \alpha^{\frac{1+\alpha}{1-\alpha}}(1-\alpha)h^* > 0,\\
\zeta_3 &\equiv \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left((1-\alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha))n^* + \alpha^{\frac{\alpha}{1-\alpha}}(1+\alpha)\right)h^* > 0,\\
\zeta_4 &\equiv \phi^{-1}(n^*)^{-1} > 0,\\
\zeta_5 &\equiv \hat{\theta}\psi\alpha^{\frac{\psi}{1-\alpha}}(1-\alpha^{\frac{1}{1-\alpha}})b^{\psi-1}\left((1-\alpha^{\frac{1}{1-\alpha}})n^* + \alpha^{\frac{1}{1-\alpha}}\right)^{\psi-1} n^*(h^*)^{\psi-1} > 0,\\
\zeta_6 &\equiv \hat{\theta}(1-\psi)\alpha^{\frac{\psi}{1-\alpha}}b^{\psi-1}\left((1-\alpha^{\frac{1}{1-\alpha}})n^* + \alpha^{\frac{1}{1-\alpha}}\right)^{\psi}(h^*)^{\psi-1} > 0.
\end{aligned}
$$

The determinant of B must be positive and determined by

$$
\det B = c^*\left(\zeta_2\zeta_5 + \zeta_4\zeta_6\right) > 0.
$$

The trace of B is equal to

$$
\begin{aligned}
\operatorname{tr} B &= -2\zeta_4 + \phi^{-1} + \zeta_3 - \zeta_1 - \zeta_6\\
&= \alpha^{\frac{2\alpha}{1-\alpha}}(1-\alpha)(1+\alpha)h^* - 2\phi^{-1}(n^*)^{-1} + \phi^{-1} - (1-\psi)\theta
\end{aligned}
$$

noting $c^* = -\zeta_4 + \phi^{-1} + \zeta_3$. To ensure the stability of the steady state, it is suffices to assume that the trace of B is strictly negative.[4] To do so, θ and $\hat{\theta}$ are normalized to be equal and sufficiently large, leading to $\frac{\theta}{\hat{\theta}} = 1$. This normalization makes the trace of B strictly negative but does not cause the loss of generality in any sense. That is, this does not change the steady state values (n^*, h^*, c^*) and condition (27) at all. This is because the steady state conditions, (24), (26), and (27), depend only on $\frac{\theta}{\hat{\theta}}$, neither solely on θ nor $\hat{\theta}$.

The above discussion provides a formal proof for Theorem 1 in Furukawa (2010).

[4]This, together with $\det B > 0$, implies that the log-linearized system has two negative and one positive eigenvalues. Since the 3×3 system has one jumpable and two non-jumpable variables, this guarantees the local saddle-path stability of the system. Note that the stability of the original system can be also guaranteed by taking into account the Local Manifold Theorem, since the steady state of the system is unique.

2 Proof for Proposition 1

Proposition 1 *There is an inverted-U relationship between the strength of IPR protection and the rate of innovation if and only if*

$$b > \alpha^{\frac{2\psi}{(1-\alpha)(1-\psi)}}(1-\alpha)(1-\psi\alpha^{-\frac{1}{1-\alpha}})/\rho\,(1-\psi)\,.$$

Otherwise, the relationship is globally upward sloping.

Proof.
Define

$$\varpi \equiv \frac{\varpi_2}{\varpi_1} = b^{-1}\alpha^{\frac{\alpha}{1-\alpha}+\frac{1}{1-\alpha}\frac{1}{1-\psi}}(1-\alpha)\left(\frac{\hat{\theta}}{\theta}\right)^{\frac{1}{1-\psi}} \text{ and } \Phi \equiv \frac{1}{\phi n^*}.$$

From (24) and (26),

$$\frac{1}{\phi} = \frac{1}{1-\alpha^{\frac{1}{1-\alpha}}}\left(\frac{\Phi+\rho}{\varpi}\right)^{\frac{1-\psi}{\psi}}\Phi - \frac{\alpha^{\frac{1}{1-\alpha}}}{1-\alpha^{\frac{1}{1-\alpha}}}\Phi \equiv f(\Phi), \tag{28}$$

$$f'(\Phi) = \frac{\varpi^{-\frac{1-\psi}{\psi}}}{1-\alpha^{\frac{1}{1-\alpha}}}\left[(\Phi+\rho)^{\frac{1-2\psi}{\psi}}\left(\frac{\Phi}{\psi}+\rho\right) - \alpha^{\frac{1}{1-\alpha}}\varpi^{\frac{1-\psi}{\psi}}\right], \tag{29}$$

$$f''(\Phi) = \frac{1-\psi}{\varpi^2\psi}\frac{\left(\frac{\Phi}{\psi}+2\rho\right)}{1-\alpha^{\frac{1}{1-\alpha}}}\left(\frac{\Phi+\rho}{\varpi}\right)^{\frac{1-\psi}{\psi}-2} > 0. \tag{30}$$

These conditions are so complex, that graphical analysis is useful. See Figure A, in which $\frac{1}{\phi}$ is measured along the vertical line and Φ is measured along the horizontal line. It depicts a graph of $f(\Phi)$ and a graph of Φ as the 45 degree line. Since

$$g^* = \Phi - \frac{1}{\phi}, \tag{31}$$

for a particular $\frac{1}{\phi} = \sigma'$, the length of a corresponding horizontal segment between $\Phi = \sigma'$ and $\Phi = f^{-1}(\sigma')$ is equal to g^* when $\phi = \frac{1}{\sigma'}$, as shown in Figure A as segment g. Note that $g^* \leq 0$ for the orange area in Figure A, where a BGP cannot exist.

Function Φ satisfies the following properties:

- f is globally strictly convex

- $f(0) = 0$
- $f(+\infty) = +\infty$
- $f'(0) = \frac{\varpi^{-\frac{1-\psi}{\psi}} \rho^{\frac{1-\psi}{\psi}} - \alpha^{\frac{1}{1-\alpha}}}{1 - \alpha^{\frac{1}{1-\alpha}}}$
- $f'(+\infty) = +\infty$[5]

The following properties are important in the following analysis:

- Points at which a graph of f intersects the horizontal line are point 0 and point $\Phi^0 \equiv \alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}} \varpi - \rho$:

$$f(\Phi) = 0 \text{ at } \Phi = \{0,\ \Phi^0\} \tag{32}$$

- A point at which a graph of f intersects the 45 degree line is point $\varpi - \rho$:

$$f(\varpi - \rho) = \Phi \tag{33}$$

- A point at which the slope of a graph of f is equal to 1 is point $\hat{\Phi}$:[6]

$$f'(\hat{\Phi}) = 1 \text{ such as } \left(\hat{\Phi} + \rho\right)^{\frac{1-2\psi}{\psi}} \left(\frac{\hat{\Phi}}{\psi} + \rho\right) = \varpi^{\frac{1-\psi}{\psi}}, \tag{34}$$

There are four possible cases depending on the value of $f'(0)$:

a1 $f'(0) \leq 0$ and $f'(\Phi^0) > 1$ (hence, $\hat{\Phi} < \Phi^0$)

a2 $f'(0) \leq 0$ and $f'(\Phi^0) < 1$ (hence, $\hat{\Phi} > \Phi^0$)

b $0 < f'(0) < 1$

c $1 \leq f'(0)$.

[5]When $0 < \phi < 0.5$, this fact can be easily shown. When $0.5 < \phi < 1$, see the following.

$$\lim_{\Phi \to \infty} \frac{\frac{\Phi}{\psi} + \rho}{(\Phi + \rho)^{\frac{2\psi - 1}{\psi}}} = \lim_{\Phi \to \infty} \frac{(\Phi + \rho)^{\frac{1-\psi}{\psi}}}{(2\psi - 1)} = +\infty.$$

[6]Note that $f'(\Phi) < 1$ for $\Phi < \hat{\Phi}$ and $f'(\Phi) \geq 1$ for $\Phi \geq \hat{\Phi}$.

Figures B describes these four cases. Then, the following three claims completely characterize the rate of innovation g^* as a function of the IPR strength ϕ. Detailed caliculations appear in footnotes.

Claim 2 *In case (c), any nontrivial equilibrium, with $g^* > 0$, does not exist. The condition for this situation is:*

$$1 \leq \frac{\rho}{\varpi}.^{7} \tag{35}$$

Claim 3 *In case (b), the relationship between g^* and ϕ^{-1} is inverted U-shaped.*[8] *The condition for this situation is:*

$$\alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}} < \frac{\rho}{\varpi} < 1.^{9} \tag{36}$$

$g^*(\phi^{-1})$ is maximized at $\phi^{-1} = f(\hat{\Phi})$ and its maximum is $\hat{\Phi} - f(\hat{\Phi})$.

Claim 4 *In case (a1) and (a2),*

$$\frac{\rho}{\varpi} \leq \alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}}. \tag{37}$$

Case (a2): The relationship between g^* and ϕ^{-1} is inverted U-shaped when $f'(\Phi^0) < 1$, which is equivalentl to

$$\frac{\rho}{\varpi} > \alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}} \frac{1 - \psi\alpha^{-\frac{1}{1-\alpha}}}{1-\psi}.^{10} \tag{38}$$

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[7] The condition for case (c), $1 < f'(0)$, can be expressed as $\varpi^{-\frac{1-\psi}{\psi}} \rho^{\frac{1-\psi}{\psi}} > 1$. This is the same as $b^{-1}\alpha^{\frac{1+\alpha-\alpha\psi}{(1-\alpha)(1-\psi)}}(1-\alpha)\left(\frac{\hat{\theta}}{\theta}\right)^{\frac{1}{1-\psi}} > \rho$ in Claim 1 as well as a condition in Theorem 1 of Furukawa (2010), $\hat{\phi} > 0$.

[8] For smaller ϕ^{-1} such that $\phi^{-1} < f(\hat{\Phi})$, $f'(\Phi) = \frac{d\phi^{-1}}{d\Phi} < 1$, noting that $\frac{dg^*}{d\phi^{-1}} = \frac{d\Phi}{d\phi^{-1}} - 1 = \frac{1}{f'(\Phi)} - 1 > 0$; g^* is an increasing function of ϕ^{-1}. See (31) and (34).

For larger ϕ^{-1} suth that $\phi^{-1} \geq f(\hat{\Phi})$, $f'(\Phi) > 1$, noting that $\frac{dg^*}{d\phi^{-1}} = \frac{d\Phi}{d\phi^{-1}} - 1 = \frac{1}{f'(\Phi)} - 1 < 0$; g^* is a decreasing function of ϕ^{-1}.

For very larger ϕ^{-1} such that $\phi^{-1} \geq \varpi - \rho$, $g^* = \Phi - \phi^{-1} \leq 0$ and any nontrivial equilibrium does not exist; $g^* = 0$. See (33).

[9] The first inequality is equivalent to the condition under which the point at which a graph of function f cuts the horizontal line is less than 0, i.e., $\alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}}\varpi - \rho < 0$. See (32).

[10] Recall that point $\Phi^0 = \alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}}\varpi - \rho$ is a point at which a graph of function f cuts the horizontal line; $f(\alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}}\varpi - \rho) = 0$.

In this case, g^* is maximized at $\phi^{-1} = f(\hat{\Phi})$ and its maximum is $\hat{\Phi} - f(\hat{\Phi})$. Case (a1): The relationship between g^* and ϕ^{-1} is downward sloping when $f'(\Phi^0) \geq 1$, which is equivalent to

$$\frac{\rho}{\varpi} \leq \alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}} \frac{1 - \psi\alpha^{-\frac{1}{1-\alpha}}}{1-\psi}.^{11} \tag{39}$$

In this case, g^* is maximized at $\phi^{-1} = 0$ and its maximum is $\Phi^0 = \alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}}\varpi - \rho$.

These claims have the following three corollaries.

Corollary 1 *The relationship between ϕ and g^* is inverted U-shaped if and only if*

$$\alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}} \frac{1 - \psi\alpha^{-\frac{1}{1-\alpha}}}{1-\psi} < \frac{\rho}{\varpi} < 1 \tag{40}$$

holds.[12][13]

Corollary 2 *If condition (40) holds (the inverted-U case), the maximum of g^* is achieved at*

$$\phi = \frac{1}{f(\hat{\Phi})}$$

and is equal to

$$\max_{\phi} g^* = \hat{\Phi} - f(\hat{\Phi}).$$

The definition of $\hat{\Phi}$ is given in (34).

Corollary 3 *If condition (40) does not hold and (39) holds (the upward sloping case), the maximum (supremum) of g^* is achieved at*

$$\phi \to +\infty$$

and is equal to

$$\max_{\phi} g^* = \alpha^{\frac{1}{1-\alpha}\frac{\psi}{1-\psi}}\varpi - \rho.$$

[11] This implies a positive relationship between ϕ and g^*.

[12] Note that the relationship between ϕ and g^* is inverted U-shaped if and only if the relationship between ϕ^{-1} and g^* is inverted U-shaped.

[13] Note that $\frac{1-\psi\alpha^{-\frac{1}{1-\alpha}}}{1-\psi} < 1$ holds since $\alpha^{-\frac{1}{1-\alpha}} > 1$. Thus, if condition (40) holds, condition (36) necessarily holds.

These immediately deliver the following proposition.

Proposition 2 *Suppose $\theta = \hat{\theta}$. Then, the condition for an inverted U, (40), can be expressed as:*

$$\frac{\alpha^{\frac{1}{1-\alpha}\frac{1+\psi}{1-\psi}+\frac{\alpha}{1-\alpha}}(1-\alpha)}{(1-\psi)\rho}\left(1-\psi\alpha^{-\frac{1}{1-\alpha}}\right) < b < \frac{\alpha^{\frac{\alpha}{1-\alpha}+\frac{1}{1-\alpha}\frac{1}{1-\psi}}(1-\alpha)}{\rho}. \tag{41}$$

The second inequality is equivalent to $\hat{\phi} > 0$, which is imposed for the existence of a BGP in Theorem 1 of Furukawa (2010).[14]

Corollary 4 *Suppose that $\alpha^{\frac{1}{1-\alpha}} < \psi$. Then, (39) does not hold and (38) always holds; the relationship between ϕ and g^* is inverted U-shaped entirely for case (c). The condition for an inverted U will be simplifed to*

$$b < \frac{\alpha^{\frac{\alpha}{1-\alpha}+\frac{1}{1-\alpha}\frac{1}{1-\psi}}(1-\alpha)}{\rho}.$$

This condition is equivalent to $\hat{\phi} > 0$, which is imposed for the existence of a BGP in Furukawa (2010).

From the above two results, Proposition 1 of Furukawa (2010) can be provided.

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[14]The formal definition of $\hat{\phi}$ is given in Furukawa (2010) as

$$\hat{\phi} = \left(b^{-1}\alpha^{\frac{1+\alpha-\alpha\psi}{(1-\alpha)(1-\psi)}}(1-\alpha)-\rho\right)^{-1}.$$

Cearly, $\hat{\phi} > 0$ is equivalent to

$$b < \frac{\alpha^{\frac{\alpha}{1-\alpha}+\frac{1}{1-\alpha}\frac{1}{1-\psi}}(1-\alpha)}{\rho}.$$

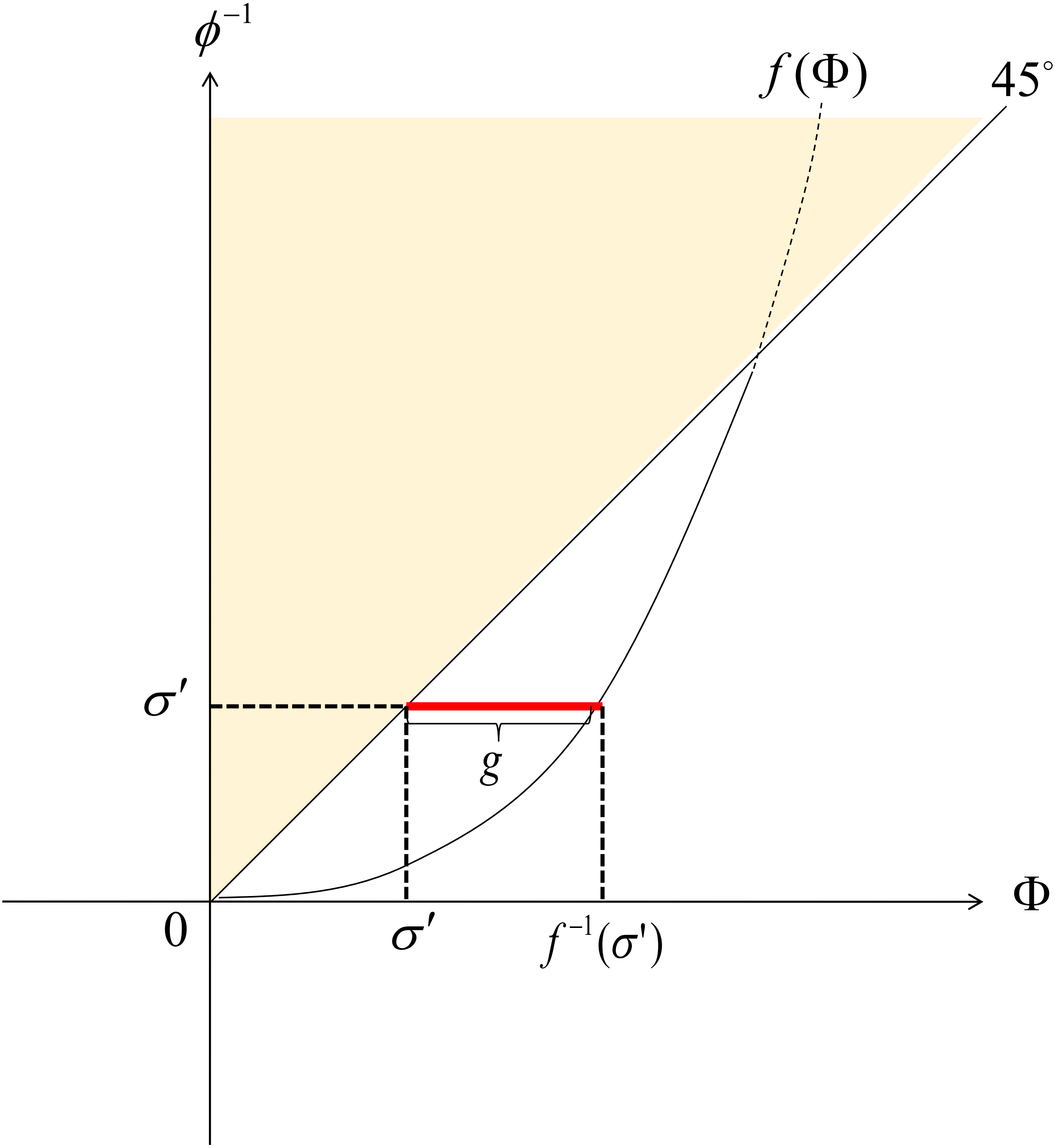

Figure A

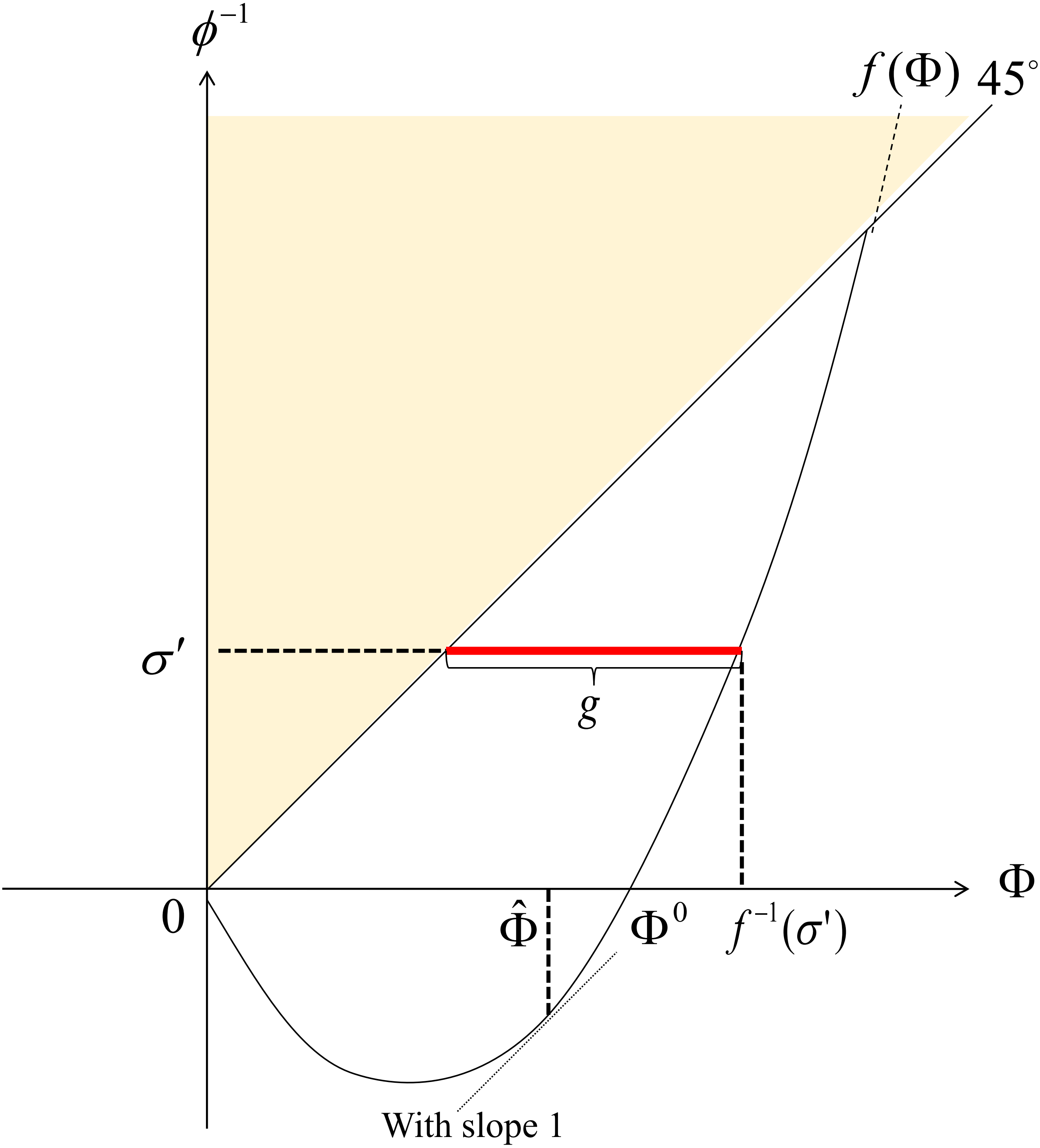

Figure B (Case a1)

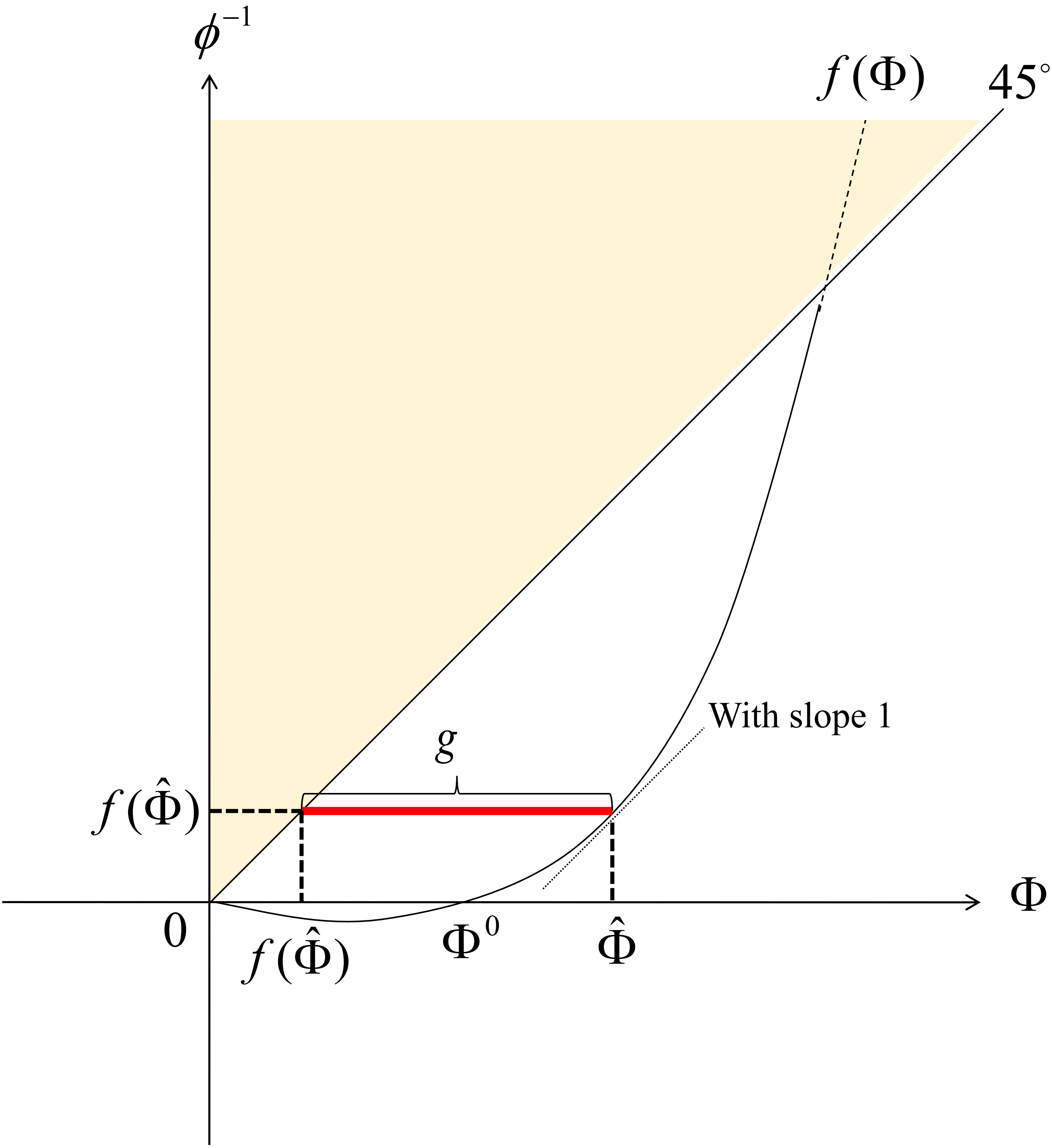

Figure B (Case a2)

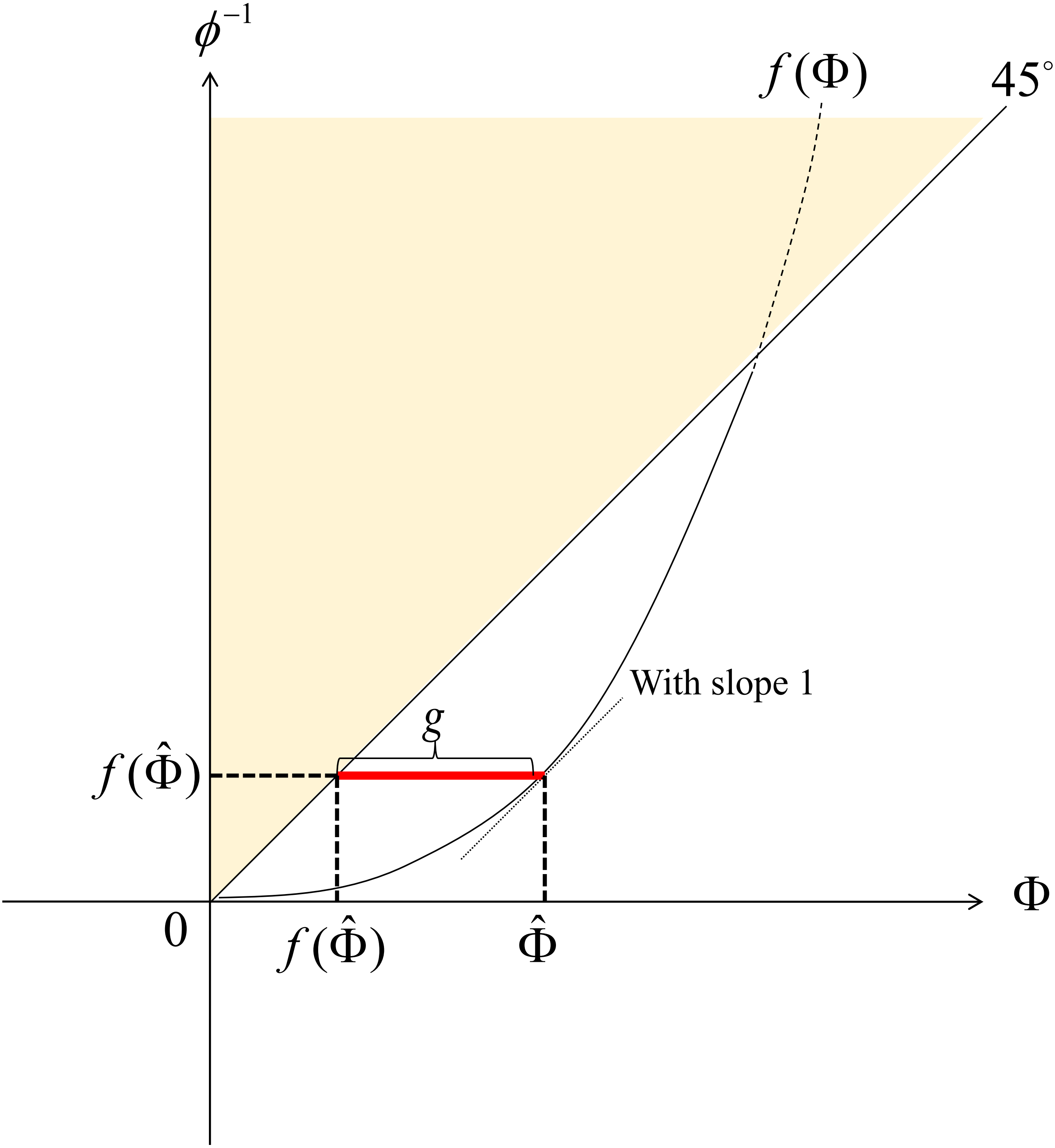

Figure B (Case b)

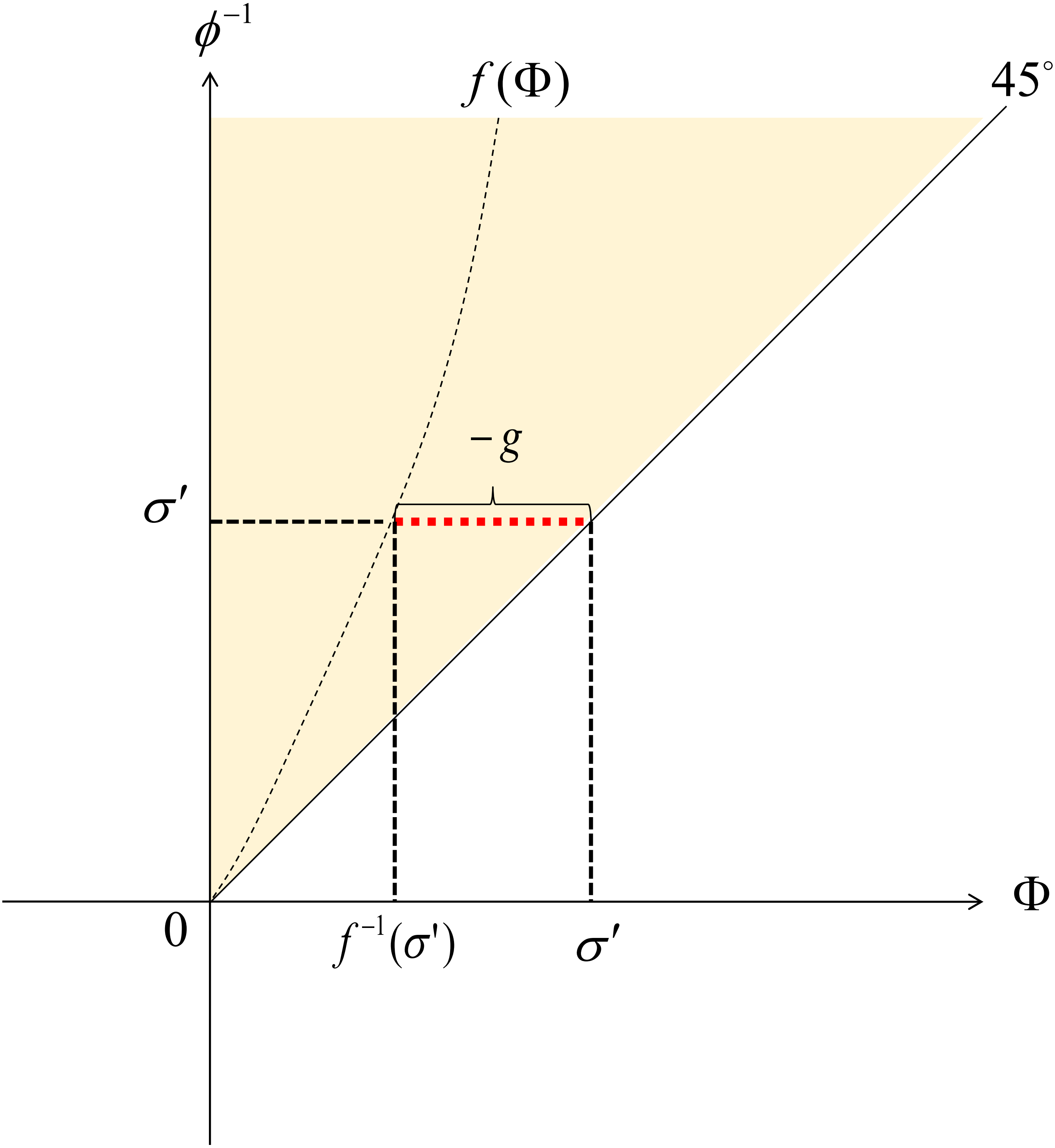

Figure B (Case c)