

Appendix to A Theory of Innovation through Global Collaboration and Wage Inequality: Not for Publication

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The mathematical details for “A Theory of Innovation through Global Collaboration and Wage Inequality” are provided in this *not-for-publication* document.

1. On the particular solution and the initial value for v_t

The dynamics of v_t follow

$$\dot{v} = A(t)v - \frac{1}{\sigma}, \quad (14)$$

where $A(t) \equiv \rho + g(\hat{n}_t^C)$ and

$$g(\hat{n}_t^C) \equiv \begin{cases} R^N + (1 - \gamma^\epsilon) \hat{n}_t^C g^C(\hat{n}_t^C) & \text{if } GR^S < \gamma^{-\epsilon} R^N \\ \hat{n}_t^C g^C(\hat{n}_t^C) & \text{if } GR^S \geq \gamma^{-\epsilon} R^N \end{cases}. \quad (15)$$

The particular solution of (14) is given by

$$v_t = e^{\int_0^t A(\tau) d\tau} \left(v_0 - \frac{1}{\sigma} \int_0^t e^{\int_0^\tau A(s) ds} d\tau \right). \quad (16)$$

Using the method of integration by parts, this expression can be rewritten as

$$v_t = \frac{1}{\sigma A(t)} + e^{\int_0^t A(\tau) d\tau} \left(v_0 + \frac{1}{\sigma} \int_0^t e^{\int_0^\tau A(s) ds} \frac{dA(\tau)/d\tau}{A(\tau)^2} d\tau - \frac{1}{\sigma A(0)} \right). \quad (17)$$

The standard transversality condition works in this model, ensuring that $\lim_{t \rightarrow \infty} v_t$ must be nonzero and bounded from above. Therefore, since $\frac{1}{\sigma A(t)} > 0$ is bounded and $\lim_{t \rightarrow \infty} e^{\int_0^t A(\tau) d\tau} = \infty$ in the right-hand side of (17), the transversality condition requires that the initial value of v_t is uniquely determined by

$$v_0 = \frac{1}{\sigma A(0)} - \frac{1}{\sigma} \int_0^\infty e^{-\int_0^\tau A(s) ds} \frac{A'(\tau)}{A(\tau)^2} d\tau. \quad (18)$$

By substituting this initial value into the particular solution (16), I obtain

$$v_t = \frac{1}{\sigma A(t)} - e^{\int_0^t A(\tau) d\tau} \frac{1}{\sigma} \int_t^{+\infty} e^{-\int_0^\tau A(s) ds} \frac{A'(\tau)}{A(\tau)^2} d\tau. \quad (19)$$

Notice that the second term in (19) converges to 0 as t goes to infinity. In order to

see this, take the limit of the second term: $\frac{1}{\sigma} \lim_{t \rightarrow \infty} \left(\frac{\int_t^{+\infty} e^{-\int_0^\tau A(s) ds} \frac{A'(\tau)}{A(\tau)^2} d\tau}{e^{-\int_0^t A(\tau) d\tau}} \right) \equiv \phi$.

Using l'Hôpital's rule, this can be rewritten as $\frac{1}{\sigma} \lim_{t \rightarrow \infty} \frac{A'(t)}{A(t)^3}$. Taking into account $A'(t) \rightarrow 0$ and $A(t) \rightarrow A(n^*)$, ϕ is equal to zero. This fact implies that as t converges to ∞ , the market value of innovations, v_t , converges to $v^* = \frac{1}{\sigma A(n^*)} = \frac{1}{\sigma(\rho+g^*)}$.

2. Proof that the static effect of an increase in R^S on w_R^S is negative.

Note that, by the proof of Theorem 1, $w_R^S = w_R^N \hat{n}^C (1 - \gamma) G^{\frac{1}{\varepsilon}}$. Thus, since \hat{n}^C is a non-jumpable variable, the *instantaneous* (short-run) effects of an increase in R^S on w_R^S and w_R^N have the same direction. Then, since $w_R^N = v$ by $V = d$, the fact that $\frac{dv}{dR^S} < 0$ holds in the short run implies that $\frac{dw_R^S}{dR^S} < 0$ holds in the short run. The following derivation proves this relation.

Recall that, by (8) and (9), the dynamics of \hat{n}^C follow

$$\frac{\dot{\hat{n}}^C}{\hat{n}^C} = GR^S (1 - (1 - \gamma^\varepsilon) \hat{n}^C) - R^N$$

for the coexistence case where $GR^S < \gamma^{-\varepsilon} R^N$. This differential equation is separable, and can be transformed to $\frac{1}{\hat{n}^C (GR^S (1 - (1 - \gamma^\varepsilon) \hat{n}^C) - R^N)} d\hat{n}^C = dt$. Given the initial condition \hat{n}_0^C , the particular solution is

$$\hat{n}_t^C = n^* \left(1 - \left(1 - \frac{n^*}{\hat{n}_0^C} \right) e^{-(GR^S - R^N)t} \right)^{-1}, \quad (20)$$

taking into account $n^* = \frac{GR^S - R^N}{GR^S (1 - \gamma^\varepsilon)}$. The particular solution for v_t is given by equation (19), in which

$$A(t) = \rho + R^N + GR^S (1 - \gamma^\varepsilon) \hat{n}_t^C \quad (21)$$

holds from (9) and (15).

The economy is assumed to be initially in the steady state, where $\hat{n}_0^C = n^* = \frac{GR^S - R^N}{GR^S(1-\gamma^\epsilon)}$ and $v_0 = v^* = \frac{1}{\sigma(\rho + GR^S)}$. When an increase in R^S occurs, the steady state value of \hat{n}_t^C changes from n^* to $n^{**} = \frac{G(R^S + \Delta) - R^N}{G(R^S + \Delta)(1-\gamma^\epsilon)} > n^*$, where Δ is a small positive number. The new transitional path, accompanied by an increase in R^S , corresponds to a path starting from the initial point n^{*12} and converging up to the new steady state n^{**} in the dynamical system ((8) and (14)) where R^S is changed to $R^S + \Delta$. From (20), such a path can therefore be formally described by the following particular solution.

$$\hat{n}_t^C = \frac{n^{**}}{1 - \left(1 - \frac{n^{**}}{n^*}\right) e^{-(G(R^S + \Delta) - R^N)t}}. \quad (22)$$

This discussion proves the trajectory of \hat{n}_t^C described in Figure 6.

From (18) and (21), a new initial value for v_t corresponding to the new path for \hat{n}_t^C following (22) is determined by the transversality condition as follows.¹³

$$v_0^{**} = \frac{1}{\sigma \hat{A}(0)} - \frac{1}{\sigma} \int_t^{+\infty} e^{-\int_0^\tau \hat{A}(s) ds} \frac{\hat{A}'(\tau)}{\hat{A}(\tau)^2} d\tau, \quad (23)$$

in which $\hat{A}(t) = \rho + R^N + G(R^S + \Delta)(1 - \gamma^\epsilon) \hat{n}_t^C$, and \hat{n}_t^C follows (22). From (23), noting $v_0 = \frac{1}{\sigma(\rho + GR^S)}$, we can derive

$$v_0^{**} - v_0 = \frac{-G\Delta}{\sigma(\rho + G(R^S + \Delta))(\rho + GR^S)} - \frac{1}{\sigma} \int_t^{+\infty} e^{-\int_0^\tau \hat{A}(s) ds} \frac{\hat{A}'(\tau)}{\hat{A}(\tau)^2} d\tau.$$

Note that $\hat{A}'(\tau) = G(R^S + \Delta)(1 - \gamma^\epsilon) \dot{\hat{n}}_t^C$ is greater than 0 because $\dot{\hat{n}}_t^C > 0$ at every date (i.e., \hat{n}_t^C monotonically converges from n^* to $n^{**} > n^*$). Thus, $v_0^{**} - v_0 < 0$ holds for any small $\Delta > 0$. This proves that an increase in R^S leads to a reduction in w_R^S *instantaneously*. The above discussion formally demonstrates that the static effect of an increase in R^S on w_R^S is negative, which guarantees that the trajectories shown in Figure 6 are adequate.

¹²Note that \hat{n}_t^C is a non-jumpable (pre-determined) variable.

¹³Note that v_t is a jumpable variable.